Modely a metódy lineárneho a celočíselného programovania

(Tézy k prenáške č. 11)

Téma prednášky

Dekompozičné prístupy pre riešenie úloh lineárneho programovania

(Časť 2 - príklad)

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Príklad č. 8.1

Riešte nasledujúcu úlohu LP dekompozičnou metódou Dantziga – Wolfa

$$f(\mathbf{x}) = 4x_1 + 5x_2 + 4x_3 + 9x_4 + 3x_5 \to \max$$

pri ohraničeniach

$$\begin{aligned}
x_1 + 2x_2 + x_3 + x_4 + 3x_5 &\le 7\\
2x_1 + x_2 + 3x_3 + x_4 + x_5 &\le 8\\
x_1 + 3x_2 &\le 6\\
2x_1 + x_2 &\le 6\\
x_3 + 2x_4 + 4x_5 &\le 6\\
x_1, x_2, x_3, x_4, x_5 &\ge 0
\end{aligned}$$

Koncern K má dve divízie D^1, D^2,

p=1,2,m_0=2, m_1=2, m_2=1, n_1=2, n_2=3,

$$\mathbf{D}^1 = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}, \mathbf{D}^2 = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}, \mathbf{c}^1 = \begin{pmatrix} 4 & 5 \end{pmatrix}, \mathbf{c}^2 = \begin{pmatrix} 4 & 9 & 3 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 3 \\ 2 & 1 & 3 & 1 & 1 \end{pmatrix}$$

$$\mathbf{b}^1 = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \mathbf{b}^2 = (6), \mathbf{b}^0 = \begin{pmatrix} 7 \\ 8 \end{pmatrix}, \mathbf{x}^1 = \begin{pmatrix} x_1 & x_2 \end{pmatrix}, x^2 = \begin{pmatrix} x_3 & x_4 & x_5 \end{pmatrix}$$

Dekompozícia

Inicializačná etapa - Pre každú divíziu skonštruujeme

$$\Omega^p = \left\{ \mathbf{x}^{pr} \middle| \mathbf{D}^p \mathbf{x}^{pr} \le \mathbf{b}^p, \mathbf{x}^{pr} \ge 0, p = 1, \ldots, k; r = 1, \ldots, q_p \right\}$$

Divízia I

$$f(x_1, x_2) = 4x_1 + 5x_2 \to \max$$

pri ohraničeniach

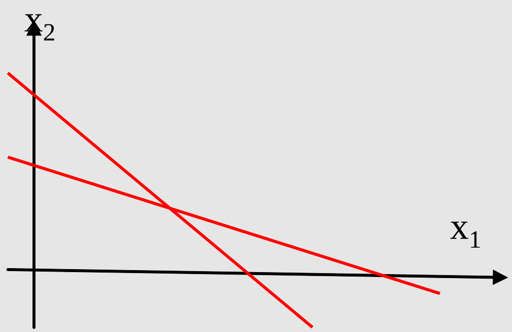
$$x_1 + 3x_2 \le 6$$
$$2x_1 + x_2 \le 6$$
$$x_1, x_2 \ge 0$$

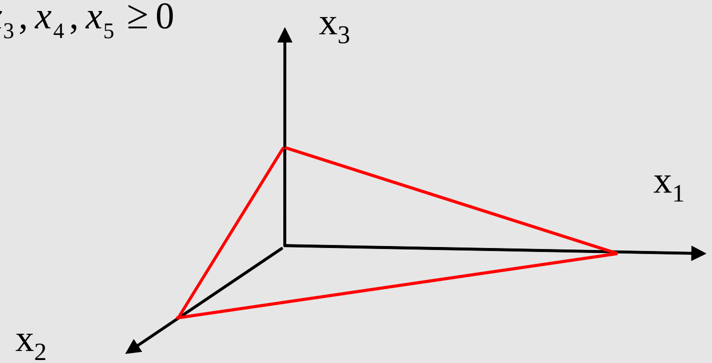
Divízia II

$$f(x_1, x_2, x_2) = 4x_3 + 9x_4 + 3x_5 \to \max$$

pri ohraničeniach

$$x_3 + 2x_4 + 4x_5 \le 6$$
$$x_3, x_4, x_5 \ge 0$$





$\mathbf{x}^{11} = (3 \quad 0)$, $\mathbf{x}^{12} = (0 \quad 2)$, $x^{21} = (0 \quad 0 \quad 3/2)$, $x^{22} = (0 \quad 3 \quad 0)$

$\mathbf{x}^{14} = (0 \quad 0)$, $\mathbf{x}^{13} = (12/5 \quad 6/5)$, $x^{23} = (6 \quad 0 \quad 0)$, $x^{24} = (0 \quad 0 \quad 0)$

Krok 2

Zaveďme substitúcie

a) $w_{pr} = \mathbf{c}^{pT}\mathbf{x}^{pr} \qquad p = 1,\ldots,k \quad r = 1,\ldots,q_p$

b) $\mathbf{P}^{pr} = \mathbf{A}^{p}\mathbf{x}^{pr} \qquad p = 1,\ldots,k \quad r = 1,\ldots,q_p$

$$w_{11} = \mathbf{c}^{1T}\mathbf{x}^{11} = \begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = 12$$

$$w_{12} = \mathbf{c}^{1T}\mathbf{x}^{12} = \begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 10$$

$$w_{21} = \mathbf{c}^{2T}\mathbf{x}^{21} = \begin{pmatrix} 4 & 9 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 3/2 \end{pmatrix} = 9/2$$

$$w_{22} = \mathbf{c}^{2T}\mathbf{x}^{22} = \begin{pmatrix} 4 & 9 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = 27$$

Krok 2

Zaveďme substitúcie

a) $w_{pr} = \mathbf{c}^{pT}\mathbf{x}^{pr} \qquad p = 1,\ldots,k \quad r = 1,\ldots,q_p$

b) $\mathbf{P}^{pr} = \mathbf{A}^{p}\mathbf{x}^{pr} \qquad p = 1,\ldots,k \quad r = 1,\ldots,q_p$

$$\mathbf{P}^{11} = \mathbf{A}^{1}\mathbf{x}^{11} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\mathbf{P}^{12} = \mathbf{A}^{1}\mathbf{x}^{12} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\mathbf{P}^{21} = \mathbf{A}^{2}\mathbf{x}^{21} = \begin{pmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 3/2 \end{pmatrix}$$

$$\mathbf{P}^{22} = \mathbf{A}^{2}\mathbf{x}^{22} = \begin{pmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Dostávame úplnú hlavnú úlohu koncernu (UHU) v tvare

$$F\left(\lambda_{pr}\right)=12\lambda_{11}+10\lambda_{12}+{}^{9}\!/_{2}\,\lambda_{21}+27\lambda_{22}\to\max$$

$$\begin{aligned}
3\lambda_{11}+4\lambda_{12}+{}^{9}\!/_{2}\,\lambda_{21}+3\lambda_{22}&\le 7\\
6\lambda_{11}+2\lambda_{12}+{}^{3}\!/_{2}\,\lambda_{21}+3\lambda_{22}&\le 8\\
\lambda_{11}+\lambda_{12}\qquad\qquad\qquad\qquad&=1\\
\lambda_{21}+\lambda_{22}&=1\\
\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}&\ge 0
\end{aligned}$$

$$F\left(\lambda_{pr}\right)=12\lambda_{11}+10\lambda_{12}+{}^{9}\!/_{2}\,\lambda_{21}+27\lambda_{22}\to\max$$

$$\begin{aligned}
3\lambda_{11}+4\lambda_{12}+{}^{9}\!/_{2}\,\lambda_{21}+3\lambda_{22}+s_1&=7\\
6\lambda_{11}+2\lambda_{12}+{}^{3}\!/_{2}\,\lambda_{21}+3\lambda_{22}+s_2&=8\\
\lambda_{11}+\lambda_{12}+w_1&=1\\
\lambda_{21}+\lambda_{22}+w_2&=1\\
\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}&\ge 0\\
s_1,s_2,w_1,w_2&\ge 0
\end{aligned}$$

Riešme úplnú hlavnú úlohu koncernu (UHU) simplexovou metódou

1. Fáza

	F(λ)	12	10	9/2	27	0	0	0	0	
	p(w)	0	0	0	0	0	0	1	1	
λ_B	c_B	λ_{11}	λ_{12}	λ_{21}	λ_{22}	s_1	s_2	w_1	w_2	b
s_1	0	3	4	9/2	3	1	0	0	0	7
s_2	0	6	2	3/2	3	0	1	0	0	8
w_1	1	1	1	0	0	0	0	1	0	1
w_2	1	0	0	1	1	0	0	0	1	1
c_j-z_j		-1	-1	-1	-1	0	0	0	0	2

2. Fáza ....

	F(λ)	12	10	9/2	27	0	0	0	0	
	p(w)	0	0	0	0	0	0	1	1	
λ_B	c_B	λ_{11}	λ_{12}	λ_{21}	λ_{22}	s_1	s_2	w_1	w_2	b
s_1	0	0	0	9/8	0	1	1/4	-9/2	-15/4	3/4
λ_{22}	27	0	0	1	1	0	0	0	1	1
λ_{11}	12	1	0	-3/8	0	0	1/4	-1/2	-3/4	3/4
λ_{12}	10	0	1	3/8	1	0	-1/4	-1/4	¾	1/4
c_j-z_j		0	0	-87/4	0	0	0	-9	-51/2	77/2

$$OR:\ \text{PU}\ \boldsymbol{\lambda}^* = (\lambda_{11}\ \ \lambda_{12}\ \ \lambda_{21}\ \ \lambda_{22}\ \ s_1\ \ s_2) = \left(3/4\ \ 1/4\ \ 0\ \ 1\ \ 3/4\ \ 0\right)$$

$$\text{DU}\ (u_1\ \ u_2\ \ v_1\ \ v_2) = (0\ \ 1/2\ \ 9\ \ 51/2)$$

Dekompozícia

Hľadá sa nepoužitá výrobná stratégia prvej divízie, ktorá najviac porušuje KO pre UHU

$$L^p = \psi(\mathbf{x}) = \left(\mathbf{c}^T - \mathbf{u}^T \mathbf{A}^p\right)\mathbf{x}^p \rightarrow \max$$

$$\mathbf{D}^p \mathbf{x}^p \leq \mathbf{b}^p$$

$$\mathbf{x}^p \geq \mathbf{0}$$

$$L^1 = \left(\begin{pmatrix} 4 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1 + 9/2x_2 \rightarrow \max$$

$$x_1 + 3x_2 \leq 6$$

$$2x_1 + x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

$$OR\ L1: \quad \mathbf{x}^* = \mathbf{x}^{13} = \begin{pmatrix} 12/5 & 6/5 \end{pmatrix} \quad L1 = 63/5$$

$$L1 - v_1 = 63/5 - 9 = 18/5 \approx\leq 0$$

Podmienka optimálnosti pre divíziu D1 nie je splnená !!!

Dekompozícia

Hľadá sa nepoužitá výrobná stratégia druhej divízie, ktorá najviac porušuje KO pre UHU

$$L^p = \psi(\mathbf{x}) = \left(\mathbf{c}^T - \mathbf{u}^T \mathbf{A}^p\right)\mathbf{x}^p \to \max$$

$$\mathbf{D}^p \mathbf{x}^p \le \mathbf{b}^p$$

$$\mathbf{x}^p \ge \mathbf{0}$$

$$L^2 = \left(\begin{pmatrix}4 & 9 & 3\end{pmatrix} - \begin{pmatrix}0 & 1/2\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\ 3 & 1 & 1\end{pmatrix}\right)\begin{pmatrix}x_3\\ x_4\\ x_5\end{pmatrix} = 5/2x_3 + 17/2x_4 + 5/2x_5 \to \max$$

$$x_3 + 2x_4 + 4x_5 \le 6$$

$$x_3, x_4, x_5 \ge 0$$

$$OR\ L2: \quad \mathbf{x}^* = \mathbf{x}^{22} = \begin{pmatrix}0 & 3 & 0\end{pmatrix} \quad L2 = 51/2$$

$$L2 - v_2 = 51/2 - 51/2 = 0 \le 0 \quad OK$$

Podmienka optimálnosti pre divíziu D2 je splnená !!!

$$L^s - v_s = \max_{p=1,2}\left(L^p - v_p\right) = \max\left\{\left(L^1 - v_1\right), \left(L^2 - v_2\right)\right\} = \max\{18/5, 0\} = 18/5 = \left(L^1 - v_1\right)$$

$s = 1, \quad \Rightarrow x^{13}$ *je nový analyzovaný krajný bod pre prvú divíziu D1*

Do UHU doplníme nový stĺpec takto:

$$\mathbf{P}^{13} = \mathbf{A}_1\mathbf{x}^{13} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 12/5 \\ 6/5 \end{pmatrix} = \begin{pmatrix} 24/5 \\ 6 \end{pmatrix} \quad w_{13} = \mathbf{c}^{1T}\mathbf{x}^{13} = \begin{pmatrix} 4 & 5 \end{pmatrix}\begin{pmatrix} 12/5 \\ 6/5 \end{pmatrix} = 78/5$$

Doplňme stĺpec o koeficienty riadkov podmienok konvexných kombinácií, dostávame

$$\tilde{\mathbf{P}}^{13} = \begin{pmatrix} 24/5 \\ 6 \\ 1 \\ 0 \end{pmatrix}$$

Rozložme stĺpec v aktuálnej báze

$$\tilde{\mathbf{P}}^{13}_B = \begin{pmatrix} 1 & 1/4 & -9/2 & -15/4 \\ 0 & 0 & 0 & 1 \\ 0 & 1/4 & -1/2 & -3/4 \\ 0 & -1/4 & -1/4 & 3/4 \end{pmatrix}\begin{pmatrix} 24/5 \\ 6 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

	F(λ)	12	10	9/2	27	0	0	0	0	78/5	
	p(w)	0	0	0	0	0	0	1	1	0	
λ_B	**c_B**	λ_{11}	λ_{12}	λ_{21}	λ_{22}	s_1	s_2	w_1	w_2	λ_{13}	b
s_1	0	0	0	9/8	0	1	1/4	-9/2	-15/4	9/5	3/4
λ_{22}	27	0	0	1	1	0	0	0	1	0	1
λ_{11}	12	1	0	-3/8	0	0	1/4	-1/2	-3/4	1	3/4
λ_{12}	10	0	1	3/8	1	0	-1/4	-1/4	¾	0	1/4
c_j-z_j		0	0	-87/4	0	0	0	-9	-51/2	18/5	77/2

Ďalší výpočet riešenia UHU simplexovou metódou a vykonanie ďalších iterácií na báze metódy Dantzigaa – Wolfa prenechávame na čitateľa .....