One Application of Recursive formula in Life Insurance

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Recursive formula is a difference equation with known starting values. Thus, if the starting value is known (usually zero), the ending value (usually maturity amount in life insurance product) and all the intermediate values can be derived easily.

In this paper we shall discuss one practical application of recursive formulas in Life Insurance Mathematics. We shall analyse the well known relationship between successive reserves for life assurance products.
The main part of the article will be the procedure for deriving a recursive formula for a traditional whole life reserve with a death benefit of 1.

1. Some basic results.

A recursive formula is the formula where the current result is generated from previous results once the starting values are given.
In this note we shall speak about special difference equation: If the starting value of the recursion is known, then we can easy derive the ending value (and also all the intermediate values).

We know this form of first order linear difference (recursive) equation.

$$x_{k+1} = a_k \cdot x_k + b_k, \quad k = 0,1,2,...$$

Define

$$d_{k+1} = a_0 \cdot a_1 ... \cdot a_k$$

If we divide both sides of original equation by $d_{k+1} = a_0 \cdot a_1 \ldots \cdot a_k$ we obtain:

$$\frac{x_{k+1}}{d_{k+1}} = \frac{x_k}{d_k} + \frac{b_k}{d_{k+1}} \qquad (1)$$

Summing previous equation from k=0 to k=t-1 yields:

$$x_t = \left(x_0 + \sum_{k=0}^{t-1} \frac{b_k}{d_{k+1}} \right) \cdot d_t$$

So once having the starting value x_0 we can find the t-th term directly without explicitly computing the intermediate terms.

There exist some examples how to apply this mathematical tool also in Life Insurance especially in the area of reserve and asset share calculations.

2. Recursive formula for whole life assurance

In this article we will derive a recursive formula for a traditional non-profit whole life assurance with a death benefit of 1.

It is well known, that the successive formula for the reserves of this type of assurance is:

$$(V_t + P)(1+i) = V_{t+1} \cdot (1 - q_t) + q_t ,$$

where

i - is the valuation interest rate,

q_t - is the probability, that a person age $x + t$ will die during one year,

P - is the net level premium payable annually in advance,

V_t and V_{t+1} - are net premium reserves at the end of the year *t* and at the end of the year *t*+1.

We arrange this formula into the form of first order linear recursive (difference) equation

$$(V_k + P)(1+i) - q_k = V_{k+1} \cdot (1-q_k)\,.$$

i. e.

$$V_{k+1} = \frac{1+i}{1-q_k} \cdot V_k + \left(P\frac{1+i}{1-q_k} - \frac{q_k}{1-q_k} \right) \qquad (2)$$

Computing the compounding element

$$d_{k+1} = \prod_{r=0}^{k} \frac{1+i}{1-q_r}$$

we can see that:

$$\frac{1}{d_k} = \frac{{}_k p_x}{1+i} = \frac{D_{x+k}}{D_x}$$

where

${}_k p_x$ is the probability that a person age *x* will live for *k* years

D_x, D_{x+k} are the commutation functions.

Now we can divide both sides of equation (2) by the discount factor d_{k+1} and we obtain the difference equation as (1) in the previous part.

After that we can sum both sides of this equation from *k*=0 to *k*=*t*-1, then generate the intermediate values and we have the well known retrospective formula for a whole life assurance:

$$V_t = P \cdot \frac{N_x - N_{x+t}}{D_{x+t}} + \frac{M_x - M_{x+t}}{D_{x+t}}$$

If we sum both sides of previous equation from issue (k=0) to maturity (k=n) we become the equation from which we can find the net level premium P.

References

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