Some Remarks On Faster Convergent Infinite Series I

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Abstract

A structure of terms of faster convergent series is studied in the paper. Necessary and sufficient conditions for the existence of faster convergent series with different types of their terms are found.

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1

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1 Introduction and preliminaries

Infinite series are important mathematical objects which can be used for solutions of mathematical as well as scientific and engineering problems (see Ferraro [15]). There are a lot of books and papers devoted to this topic (see Bromwich [9], Knopp [21]). Some of them are devoted to the study of the faster convergence of sequences, particularly to the acceleration of convergence of sequence of partial sums of series via linear and nonlinear transformations (see Bornemann, Laurie, Wagon, and Waldvogel [2], Brezinski [3; 4; 5; 6; 7], Brezinski and Redivo Zaglia [8], Cuyt and Wuytack [12], Delahaye [14], Liem, Lü, and Shih [22], Marchuk, and Shaidurov [23], Salzer [25], Sidi [26], Walz [31], Wimp [33], [34], Caliceti, Meyer-Hermann, Ribeca, Surzhykov and Jentschura [10], Homeier [17], Weniger [32]). The speed of convergence of sequences is of the central importance in the theory of sequence transformation. A sequence transformation T

$$T:\ \{s_n\}_{n=1}^{\infty} \mapsto \{s_n^\star\}_{n=1}^{\infty}$$

is a function which maps a slowly convergent sequence $\{s_n\}_{n=1}^{\infty}$ to another sequence $\{s_n^\star\}_{n=1}^{\infty}$ with better numerical properties. If $\lim\limits_{n\to\infty} s_n = s$ and $\lim\limits_{n\to\infty} s_n^\star = s^\star$ and r_n, $r_n^\star$ are truncation errors according to $s_n = s + r_n$, $s_n^\star = s^\star + r_n^\star$ then we say that the sequence $\{s_n^\star\}_{n=1}^{\infty}$ converges more rapidly to its limit $s^\star$ than the sequence $\{s_n\}_{n=1}^{\infty}$ to its limit s if

$$\lim_{n\to\infty} \frac{s_n^\star - s^\star}{s_n - s} = \lim_{n\to\infty} \frac{r_n^\star}{r_n} = 0. \tag{a}$$

We can assume that $s = s^\star$. If $\{s_n(a)\}_{n=1}^{\infty}$, $\{s_n^\star(a)\}_{n=1}^{\infty}$ are sequences of the partial sums of infinite series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} a_n^\star$ then (a) translates to

$$\lim_{n\to\infty} \frac{\sum\limits_{j=n+1}^{\infty} a_j^\star}{\sum\limits_{j=n+1}^{\infty} a_j} = 0 \tag{b}$$

and we can say similarly that $\sum\limits_{n=1}^{\infty} a_n^\star$ converges more rapidly than $\sum\limits_{n=1}^{\infty} a_n$ if (b) is satisfied. So, the rate of convergence of the infinite series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} a_n^\star$ is measured by the rate, with which its truncation errors $\sum\limits_{j=n+1}^{\infty} a_j$ and $\sum\limits_{j=n+1}^{\infty} a_j^\star$ vanish as $n \to \infty$. However the condition (b) of acceleration of the convergence is not very convenient in practical applications. It is much more convenient if either condition (a) or (b) is replaced by the condition:

$$\lim_{n\to\infty} \frac{\triangle s_n^\star(a)}{\triangle s_n(a)} = 0, \tag{c}$$

where $\triangle s_n = s_n - s_{n-1}$, $\triangle s_n^\star = s_n^\star - s_{n-1}^\star$, (see Bromwich [9], Wimp [11]).

The condition (c) has been studied in [16]. In [16] we proved the following results which extend some of [29]:
Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be convergent infinite series with positive terms and $s(a) = \lim\limits_{n\to\infty} s_n(a)$, $s(b) = \lim\limits_{n\to\infty} s_n(b)$. If there is $\lim\limits_{n\to\infty} \frac{\triangle s_n(a)}{\triangle s_n(b)}$ then

$$\lim_{n\to\infty} \frac{\triangle s_n(a)}{\triangle s_n(b)} = 0 \text{ iff } \lim_{n\to\infty} \frac{s(a) - s_{n-1}(a)}{s(b) - s_{n-1}(b)} = 0.$$

Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be convergent infinite series with nonzero terms and such that $s(b) - s_{n-1}(b) \neq 0$, $n \in N$. If

$$\lim_{n\to\infty} \frac{s(a) - s_{n-1}(a)}{s(b) - s_{n-1}(b)} = 0 \text{ then } \liminf_{n\to\infty} \frac{\triangle s_n(a)}{\triangle s_n(b)} = 0.$$

In [16] we also found conditions for convergent infinite series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ with nonzero terms for which the condition $\lim\limits_{n\to\infty} \frac{\triangle s_n(a)}{\triangle s_n(b)} = 0$ is

$a)$ necessary
$b)$sufficient
$c)$ necessary and sufficient,

so that $\lim\limits_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = 0$. Using these results we also proved in [16] that the Kummer's transformations of a wide class of convergent infinite series are faster convergent infinite series.

We recall the definition of Kummer's transformation [9]. Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} c_n$ be two convergent nonzero series such that $\lim\limits_{n\to\infty} \frac{\triangle s_n(a)}{\triangle s_n(c)} = \gamma \neq 0$. The series $\sum\limits_{n=1}^{\infty} b_n$ such that $b_1 = a_1 + \gamma(s(c) - c_1)$, $b_n = \left(1 - \gamma \frac{\triangle s_n(c)}{\triangle s_n(a)}\right) \triangle s_n(a)$, $n \geq 2$ is called Kummer's series. The rule which maps convergent infinite series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} c_n$ to infinite convergent series $\sum\limits_{n=1}^{\infty} b_n$ is called Kummer's transformation. This transformation is very convenient for practical applications. If $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series with an unknown sum $s(a)$, $\sum\limits_{n=1}^{\infty} c_n$ is a convergent series with a known sum $s(c)$ and such that $\lim\limits_{n\to\infty} \frac{\triangle s_n(a)}{\triangle s_n(c)} \neq 0$ then the Kummer's series

$\sum_{n=1}^{\infty} b_n$ has the sum $s(a)$ and

$$\lim_{n\to\infty} \frac{\triangle s_n(b)}{\triangle s_n(a)} = 0.$$

If the condition (c) ($\lim_{n\to\infty} \frac{\triangle s_n(b)}{\triangle s_n(a)} = 0$) implies faster convergence $\sum_{n=1}^{\infty} b_n$ than $\sum_{n=1}^{\infty} a_n$, then for the practical calculation of the sum $s(a)$, the Kummer's series $\sum_{n=1}^{\infty} b_n$ is better than $\sum_{n=1}^{\infty} a_n$. We also showed that the condition (c) does not generally imply the faster convergence. in the paper [16] we also proved that Kummer's series are faster convergent under much more general conditions than $a_n > 0$, $b_n > 0$, $c_n > 0$, $\gamma > 0$ as it was shown in [29].

Definition 1 *[2] Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent series such that $s(b)-s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} a_n$ is called faster convergent series than $\sum_{n=1}^{\infty} b_n$ if $\lim_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = 0$.*

We will write "fcst" instead of "faster convergent series than".

Lemma 2 *[10] Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent series with positive terms. If $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$ then $\sum_{n=1}^{\infty} a_n$ is fcst $\sum_{n=1}^{\infty} b_n$.*

Lemma 3 *[4] Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent real series. Let $s(b) - s_{n-1}(b) \neq 0$ for all $n \in \mathbb{N}$. Let $l_i(a) = \liminf_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right|$, $l_s(a) = \limsup_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right|$, $l_i(b) = \liminf_{n\to\infty} \left|\frac{s(b)-s_{n-1}(b)}{s_n(b)-s_{n-1}(b)}\right|$, $l_s(b) = \limsup_{n\to\infty} \left|\frac{s(b)-s_{n-1}(b)}{s_n(b)-s_{n-1}(b)}\right|$. Then*

(a) if $l_s(a) < \infty$, $l_i(b) > 0$ and $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$, then $\sum_{n=1}^{\infty} a_n$ is fcst $\sum_{n=1}^{\infty} b_n$

(b) if $s(a) - s_{n-1}(a) \neq 0$ for all $n \in \mathbb{N}$, $l_i(a) > 0$, $l_s(b) < \infty$ and $\sum_{n=1}^{\infty} a_n$ is fcst $\sum_{n=1}^{\infty} b_n$, then $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$.

2 Main results

The following example shows that there exist series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ such that $s(a) - s_{n-1}(a) \neq 0$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_n$ is fcst $\sum_{n=1}^{\infty} b_n$ and

$\lim\limits_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} \neq 0$.

Example 4 *Denote* $s = \sum\limits_{n=1}^{\infty} \alpha_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n q}{\sqrt{n}}$, *where* $q \in \mathbb{R} \setminus \{0\}$ *such that* $q \sum\limits_{j=n}^{\infty} \frac{(-1)^j}{\sqrt{j}} \neq -1$, $q\frac{(-1)^n}{\sqrt{n}} \neq 1$, $n \in \mathbb{N}$. *Let* $\{c_n\}_{n=1}^{\infty}$, $\{d_n\}_{n=1}^{\infty}$ *be such that* $c_1 = sx$, $c_{n+1} = c_n - \alpha_n x$, $d_n = x + c_{n+1}$, $n \in \mathbb{N}$, $x \in \mathbb{R} \setminus \{0\}$. *Then* $\lim\limits_{n\to\infty} c_n = 0$, $c_n \neq 0$, $c_{n+1} \neq c_n$, $d_n \neq 0$, $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} d_n = x$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. It is easy to see that* $\lim\limits_{n\to\infty} \prod\limits_{j=1}^{n} (1 - x_j) = 0$ *if and only if* $\sum\limits_{n=1}^{\infty} \ln|1 - x_n| = -\infty$. *From Taylor's series for* $\ln(1-x)$ *at* $x = 0$ *we have that* $\ln(1-x) < -x - \frac{x^2}{8}$ *for* $0 < |x| < 1$. *Hence* $\sum\limits_{n=1}^{\infty} \ln|1-\alpha_n| = -\infty$ *and so* $\lim\limits_{n\to\infty} \prod\limits_{j=1}^{n} (1-\alpha_j) = 0$. *Let* $B_1 \in \mathbb{R}$, $B_1 \neq 0$. *Denote* $B_{n+1} = (1-\alpha_n)B_n$ *for* $n \in \mathbb{N}$. *Then* $\lim\limits_{n\to\infty} B_n = 0$ *and* $B_n \neq B_{n+1}$, $B_n \neq 0$, $n \in \mathbb{N}$. *From* $B_{n+1} = (\frac{d_n - c_n}{d_n - c_{n+1}})B_n$ *we obtain* $d_n = \frac{c_n B_n - c_{n+1} B_{n+1}}{B_n - B_{n+1}}$. *Put* $A_n = c_n B_n$, $n \in \mathbb{N}$. *Then* $A_n \neq 0$, $A_{n+1} \neq A_n$, $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} A_n = 0$, $\lim\limits_{n\to\infty} \frac{A_n}{B_n} = 0$, $\lim\limits_{n\to\infty} \frac{A_n - A_{n+1}}{B_n - B_{n+1}} = \lim\limits_{n\to\infty} d_n = x \neq 0$. *Put* $a_n = A_n - A_{n+1}$, $b_n = B_n - B_{n+1}$, $n \in \mathbb{N}$. *We obtain convergent series* $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ *with nonzero terms such that* $s(a) - s_{n-1}(a) \neq 0$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = \lim\limits_{n\to\infty} \frac{A_n}{B_n} = 0$, $\lim\limits_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = x \neq 0$.

Example 5 *Put in the example 4* $s = \sum\limits_{n=1}^{\infty} \frac{(-1)^n q}{\sqrt[4]{n}}$, *where* $q \in \mathbb{R} \setminus \{0\}$ *such that* $\sum\limits_{j=n+1}^{\infty} \frac{(-1)^{j+1} q}{\sqrt[4]{j}} \neq p\sqrt[4]{n}$, $\frac{(-1)^{n+1} q}{\sqrt[4]{n}} \neq p\sqrt[4]{n}$, $n \in \mathbb{N}$, $c_1 = s$, $c_{n+1} = c_n - \alpha_n \sqrt[4]{n}$, $\alpha_n = \frac{(-1)^n q}{\sqrt{n}}$, $d_n = p\sqrt[4]{n} + c_{n+1}$, $B_{n+1} = (1 - p\alpha_n)B_n$, $n \in \mathbb{N}$ *where* $p = 1$ *or* $p = -1$. *We obtain convergent series* $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ *with nonzero terms such that* $s(a) - s_{n-1}(a) \neq 0$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = 0$, $\lim\limits_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = +\infty$ *if* $p = 1$ *or* $\lim\limits_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = -\infty$ *if* $p = -1$.

Lemma 6 *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be fcst* $\sum\limits_{n=1}^{\infty} b_n$ *such that* $\lim\limits_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$ *and let* $\sum\limits_{n=1}^{\infty} c_n$ *be fcst* $\sum\limits_{n=1}^{\infty} b_n$ *such that* $\lim\limits_{n\to\infty} \frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)} \neq 0$ *or* $\lim\limits_{n\to\infty} \frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}$ *does not exist and* $\liminf\limits_{n\to\infty} \left|\frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}\right| > 0$. *If* $\limsup\limits_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right| < \infty$, $\liminf\limits_{n\to\infty} \left|\frac{s(c)-s_{n-1}(c)}{s_n(c)-s_{n-1}(c)}\right| > 0$ *then* $\sum\limits_{n=1}^{\infty} a_n$ *is fcst* $\sum\limits_{n=1}^{\infty} c_n$.

Proof. It follows from Lemma 3. □

It is easy to show that there exist series $\sum_{n=1}^{\infty} c_n$, $\sum_{n=1}^{\infty} b_n$, $s(c) - s_{n-1}(c) \neq 0$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} c_n$ is fcst $\sum_{n=1}^{\infty} b_n$, $\lim_{n\to\infty} \frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}$ does not exist, $\liminf_{n\to\infty} \left|\frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}\right| > 0$ and $\liminf_{n\to\infty} \left|\frac{s(c)-s_{n-1}(c)}{s_n(c)-s_{n-1}(c)}\right| > 0$.

Lemma 7 *Let* $\sum_{n=1}^{\infty} b_n$ *be a convergent series such that* $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$. *Then there exists* $\sum_{n=1}^{\infty} a_n$ *fcst* $\sum_{n=1}^{\infty} b_n$ *such that* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$ *and* $\limsup_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right| < \infty$.

Proof. Put $B_n = s(b) - s_{n-1}(b)$, $\gamma_n = \frac{B_n}{B_n - B_{n+1}}$, $\delta_n = \frac{1}{\gamma_n}$, $n \in \mathbb{N}$. By the induction we construct a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\varepsilon_{n+1} \neq 0$, $\varepsilon_n B_n \neq \varepsilon_{n+1} B_{n+1}$, $|\varepsilon_n| \leq (c-1)|\varepsilon_n - \varepsilon_{n+1}|$, $|\delta_n \varepsilon_{n+1}| < \frac{1}{c}|\varepsilon_n - \varepsilon_{n+1}|$, $|\varepsilon_{n+1}| < \frac{|\varepsilon_n|}{2}$, $|\varepsilon_{n+1}| < \frac{|\delta_{n+1}|}{n+1}$, $n \in \mathbb{N}$, where $c > 2$, $c \in \mathbb{R}$. Let $0 < |\varepsilon_1| < |\delta_1|$. The continuity of $f_1(x) = (c-1)|\varepsilon_1 - x| - |\varepsilon_1|$ and $g_1(x) = \frac{1}{c}|\varepsilon_1 - x| - |\delta_1 x|$ implies the existence of a neighbourhood U of $x = 0$ such that $|\varepsilon_1| < (c-1)|\varepsilon_1 - x|$, $|\delta_1 x| < \frac{1}{c}|\varepsilon_1 - x|$ for $x \in U$. It implies the existence $\varepsilon_2 \neq 0$ such that $\varepsilon_1 B_1 \neq \varepsilon_2 B_2$, $|\varepsilon_1| < (c-1)|\varepsilon_1 - \varepsilon_2|$, $|\delta_1 \varepsilon_2| < \frac{1}{c}|\varepsilon_1 - \varepsilon_2|$, $|\varepsilon_2| < \frac{|\varepsilon_1|}{2}$, $|\varepsilon_2| < \frac{|\delta_2|}{2}$. Let $\varepsilon_1, ..., \varepsilon_n$ be such that $\varepsilon_j B_j \neq \varepsilon_{j+1} B_{j+1}$, $|\varepsilon_j| < (c-1)|\varepsilon_j - \varepsilon_{j+1}|$, $|\delta_j \varepsilon_{j+1}| < \frac{1}{c}|\varepsilon_j - \varepsilon_{j+1}|$, $|\varepsilon_{j+1}| < \frac{|\varepsilon_j|}{2}$, $|\varepsilon_{j+1}| < \frac{|\delta_{j+1}|}{j+1}$, $j = 1, ..., n-1$. The continuity of $f_n(x) = (c-1)|\varepsilon_n - x| - |\varepsilon_n|$ and $g_n(x) = \frac{1}{c}|\varepsilon_n - x| - |\delta_n x|$ implies that there exists $\varepsilon_{n+1} \neq 0$ such that $\varepsilon_n B_n \neq \varepsilon_{n+1} B_{n+1}$, $|\varepsilon_n| < (c-1)|\varepsilon_n - \varepsilon_{n+1}|$, $|\delta_n \varepsilon_{n+1}| < \frac{1}{c}|\varepsilon_n - \varepsilon_{n+1}|$, $|\varepsilon_{n+1}| < \frac{|\varepsilon_n|}{2}$, $|\varepsilon_{n+1}| < \frac{|\delta_{n+1}|}{n+1}$. Put $A_n = \varepsilon_n B_n$, $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$. Then $A_n \neq 0$, $A_n \neq A_{n+1}$, $n \in \mathbb{N}$ and since $\lim_{n\to\infty} A_n = 0$, $\sum_{n=1}^{\infty} a_n$ is a convergent series. From $\lim_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = \lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \varepsilon_n = 0$ it follows that $\sum_{n=1}^{\infty} a_n$ is fcst $\sum_{n=1}^{\infty} b_n$.
From inequalities
$|\varepsilon_n - \varepsilon_{n+1}||\gamma_n| \leq (|\varepsilon_n| + |\varepsilon_{n+1}|)|\gamma_n| \leq (|\varepsilon_n| + \frac{|\varepsilon_n|}{2})|\gamma_n| < (\frac{|\delta_n|}{n} + \frac{|\delta_n|}{2n})|\gamma_n| = \frac{3}{2n}$ we have $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = \lim_{n\to\infty} \frac{A_n - A_{n+1}}{B_n - B_{n+1}} = \lim_{n\to\infty} ((\varepsilon_n - \varepsilon_{n+1})\frac{B_n}{B_n - B_{n+1}} + \varepsilon_{n+1}) = 0$. Since $\frac{1}{c}|\varepsilon_n| < \frac{c-1}{c}|\varepsilon_n - \varepsilon_{n+1}| = |\varepsilon_n - \varepsilon_{n+1}| - \frac{1}{c}|\varepsilon_n - \varepsilon_{n+1}| < |\varepsilon_n - \varepsilon_{n+1}| - |\delta_n \varepsilon_{n+1}| \leq |\varepsilon_n - \varepsilon_{n+1} + \delta_n \varepsilon_{n+1}| = \left|\frac{B_n \varepsilon_n - B_{n+1}\varepsilon_{n+1}}{B_n}\right| = \left|\frac{s_n(a)-s_{n-1}(a)}{B_n}\right|$ we have that $\left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right| = \left|\frac{\varepsilon_n}{\varepsilon_n - \varepsilon_{n+1} + \delta_n \varepsilon_{n+1}}\right| \leq c$, $n \in \mathbb{N}$. □

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